3. SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

3.1 Definition of algebraic and transcendental equation.
3.2 Methods for the solution of Algebraic and Transcendental equation.
3.1 Definition of algebraic and transcendental equation.

Definition.
Examples.
3.2. Method of Bisection

A Explanation of the Method.
A Example. Compute one root of $e^x - 3x = 0$ correct to two decimal places.

Solution: - Let $f(x) = e^x - 3x$, $f(1.5) = -0.02$, $f(1.6) = 0.15$ so one root of $f(x) = 0$ lies between 1.5 and 1.6. Here $a_0 = 1.5, b_0 = 1.6$. In fourth step $a_n, b_n$ are equal up to two decimal places.
$$x_{n+1} = \frac{a_n + b_n}{2}$$

<table>
<thead>
<tr>
<th>(n)</th>
<th>(a_n)</th>
<th>(b_n)</th>
<th>(f(x_{n+1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.5</td>
<td>1.6</td>
<td>0.06</td>
</tr>
<tr>
<td>1</td>
<td>1.5</td>
<td>1.55</td>
<td>0.02</td>
</tr>
<tr>
<td>2</td>
<td>1.5</td>
<td>1.525</td>
<td>0.00056</td>
</tr>
<tr>
<td>3</td>
<td>1.5</td>
<td>1.5125</td>
<td>-0.00904</td>
</tr>
<tr>
<td>4</td>
<td>1.5062</td>
<td>1.5125</td>
<td>-0.00426</td>
</tr>
<tr>
<td>5</td>
<td>1.50935</td>
<td>1.5125</td>
<td>-0.00184</td>
</tr>
</tbody>
</table>
Thus $x=1.51$ is the root of $f(x)=0$ correct to two decimal places.
Regula-Falsi Method

Explanation of the Method
Example
The Iteration Methods

• The Iteration Method

• Aitken’s Method

• Newton-Raphson Method
The Iteration Method

Example: Find a real root of \( \cos x = 3x - 1 \).

Solution: \( f(x) = \cos x - 3x + 1 = 0 \) and \( f(0)f(-) < 0 \) so a root lies between 0 and -.

It can be written as \( x = -(1 + \cos x) \) so

\[
\phi(x) = - \frac{\phi'(x)}{\phi'(x)} \quad \phi'(x) = -1 \quad |\phi'(x)| < 1 \quad \text{in} \quad (0, -)
\]

means convergence condition is satisfied hence iteration method can be applied.
Let \( x_0 = 0 \), \( x_1 = \emptyset \), \( (x_0) = 0.66667 \),
\( x_2 = \emptyset(x_1) = 0.59530 \),
\( x_3 = \emptyset(x_2) = 0.60933 \), \( x_4 = 0.60668 \),
\( x_5 = 0.60718 \), \( x_6 = 0.60710 \), \( x_7 = 0.60710 \).
So the correct root of the equation is 0.60710 correct to five decimal places.
Aitken’s Method

Let $x = \alpha$ be a correct root of the equation $f(x) = 0$ and $I$ be an interval containing the point $x = \alpha$. The equation can be written as $x = \emptyset(x)$ such that $\emptyset(x)$ and $\emptyset'(x)$ are continuous in $|\emptyset'(x)| < 1$ for all $x$ in $I$. Let $x_{i-1}, x_i$ and $x_{i+1}$ be three successive approximations of the desired root $\alpha$. 
Continued

\[ \alpha x_i = \lambda (\alpha x_{i-1}) \] and
\[ \alpha x_{i+1} = \lambda (\alpha x_i) \] (\( \lambda \) is a constant such that \( |\varphi'(x)| \lambda < 1 \) for all \( i \)) dividing we get

\[ \frac{\alpha}{\alpha} \quad \frac{\alpha}{\alpha} \quad \Rightarrow \quad \alpha x_{i+1} - \Delta \]

\[ \Rightarrow \alpha x_{i+1} - \frac{\Delta}{\Delta} \]

where \( \Delta x_i = x_{i+1} - x_i \) and \( \Delta^2 x_{i-1} = (E-1)^2 x_{i-1} \)
Example

Solve \( \cos x = 3x - 1 \) by Aitken’s method.

Solution

Let \( x_0 = 0 \) be the initial approximation of the root. \( X_1 = x = \emptyset (x_0) = 0.6667 \),
\( x_2 = \emptyset (x_1) = 0.5953 \),
\( x_3 = \emptyset (x_2) = 0.6093 \).

Hence \( x_4 = x_3 - \frac{\Delta}{\Delta} = 0.6071 \).

Hence required root is 0.607.
Newton-Raphson Method

Let \( x = x_0 \) be an approximate value of one root of the equation \( f(x) = 0 \). If \( x = x_1 \) is the exact root then \( f(x_1) = 0 \) \hspace{1cm} (i)

\[ x_1 = x_0 + h \] \hspace{1cm} (ii)

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \] is called Newton-Raphson method.
Example

Using Newton Raphson method find the correct root of the equation $x^3 - 6x + 4 = 0$ between 0 and 1.

**Solution** $f(x) = x^3 - 6x + 4 = 0$ and $f(0)f(1) < 0$ so a root lies between 0 and 1. The value of the root is nearer to 1. Let $x_0 = 0.7$ be an approximate value of the root.

By this method, we get $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.7316$,

$x_2 = 0.7321$, $x_3 = 0.7321$, $x_4 = 0.7321$

so root of the given equation is

$= 0.7321$ (approximately).
Convergence Of Newton’s Raphson’s Method

\[ \Delta \left| f(x) \right| < \text{, where notations have their usual meaning.} \]